Monte Carlo Numerical Estimation of Hypershpere Volume

Created using Maple 14.01

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```
> restart;
  with(stats) :
   with(plots) :
   with(Statistics) :
   with(StringTools) :
   FormatTime("%m−%d−%Y, %H:%M");
```
$$\text{"03-21-2013, 21:57"} \tag{1}$$

In all Monte Carlo simulations it is necessary to generate random or pseudo-random numbers. The following statement will generate a random number drawn from a uniform distribution between 0 and 1.

```
> X := x→stats[random, uniform[0, 1]](1) :
  X( );
```
$$0.1931398164 \tag{2}$$

This tutorial will attempt to numerically determine the volume of a sphere of arbitrary dimension using the *Hit & Miss* method. This is equivalent to evaluating a *d*-dimensional integral. In *d*-dimensions, a hypersphere is defined by x1^2+x^2+x3^3+...+xd^2 < R^2.

To find the volume of a sphere we'll generate random numbers over some "square" volume and count how many land inside the sphere. The probability of the randomly generated point landing within the sphere is simple $p = V$sphere $/ V$square. If we use spheres of radius 1, then the volume of the sqaure that contains the sphere is 2^d where *d* is the number of dimensions that we're working with. The Monte Carlo simulation is used to determine $p = Zn / n$ where *n* is the number of trials and *Zn* is the number of "Hits". Collecting all of these results we have $p = Zn / n = V$sphere $/ 2$^d or *V*sphere = 2^d (*Zn* / *n*).

We'll start with the trivial example of a 1-D sphere. In 1-D, x1^2<R^2 simply translates to -R < x1 < + R. Throughout this tutorial will restrict ourselve to the positive quadrant. That's we'll generate random values of x1 between [0, R] (where R = 1 in this tutorial). The 1-D sphere is trivial because ALL of the randomly generated numbers will fall within the sphere. The volume of the 1-D sphere is just the length of the line. V1D = 2R.

```
> d := 1 :
  n := 100000 :
  Zn := 0 :
   for i from 1 to n do:
  x1 := X( ) :
  R := sqrt(x1²) :
     if R ≤ 1 then;
        Zn := Zn + 1 :
     end if;
  end do:
```
$$V1D := evalf\left(\frac{2^d \cdot Zn}{n}\right);$$
$$evalf\left(\frac{V1D}{2}\right);$$

$$V1D := 2.$$
$$1.000000000 \qquad (3)$$

> $d := 2$:
$n := 100000$:
$Zn := 0$:
for i **from** 1 **to** n **do**:
$x1 := X(\ )$:
$x2 := X(\ )$:
$R := \text{sqrt}(x1^2 + x2^2)$:
   if $R \leq 1$ **then**;
     $Zn := Zn + 1$:
   end if;
end do:

$$V2D := evalf\left(\frac{2^d \cdot Zn}{n}\right);$$

$$evalf\left(\frac{V2D}{Pi}\right);$$

$$V2D := 3.139000000$$
$$0.9991747325 \qquad (4)$$

> $d := 3$:
$n := 100000$:
$Zn := 0$:
for i **from** 1 **to** n **do**:
$x1 := X(\ )$:
$x2 := X(\ )$:
$x3 := X(\ )$:
$R := \text{sqrt}(x1^2 + x2^2 + x3^2)$:
   if $R \leq 1$ **then**;
     $Zn := Zn + 1$:
   end if;
end do:

$$V3D := evalf\left(\frac{2^d \cdot Zn}{n}\right);$$

$$evalf\left(\frac{V3D}{\frac{4}{3} Pi}\right);$$

$$V3D := 4.193760000$$
$$1.001186451 \qquad (5)$$

```
> d := 4 :
  n := 100000 :
  Zn := 0 :
  for i from 1 to n do:
  x1 := X( ) :
  x2 := X( ) :
  x3 := X( ) :
  x4 := X( ) :
  R := sqrt(x1² + x2² + x3² + x4²) :
      if R ≤ 1 then;
          Zn := Zn + 1 :
      end if;
  end do:
```

$$V4D := evalf\left(\frac{2^d \cdot Zn}{n}\right);$$

$$evalf\left(\frac{V4D}{\frac{1}{2}\,\pi^2}\right);$$

$$V4D := 4.904960000$$

$$0.9939527054 \tag{6}$$

```
> d := 5 :
  n := 100000 :
  Zn := 0 :
  for i from 1 to n do:
  x1 := X( ) :
  x2 := X( ) :
  x3 := X( ) :
  x4 := X( ) :
  x5 := X( ) :
  R := sqrt(x1² + x2² + x3² + x4² + x5²) :
      if R ≤ 1 then;
          Zn := Zn + 1 :
      end if;
  end do:
```

$$V5D := evalf\left(\frac{2^d \cdot Zn}{n}\right);$$

$$evalf\left(\frac{V5D}{\frac{8}{15}\,\pi^2}\right);$$

$$V5D := 5.226560000$$
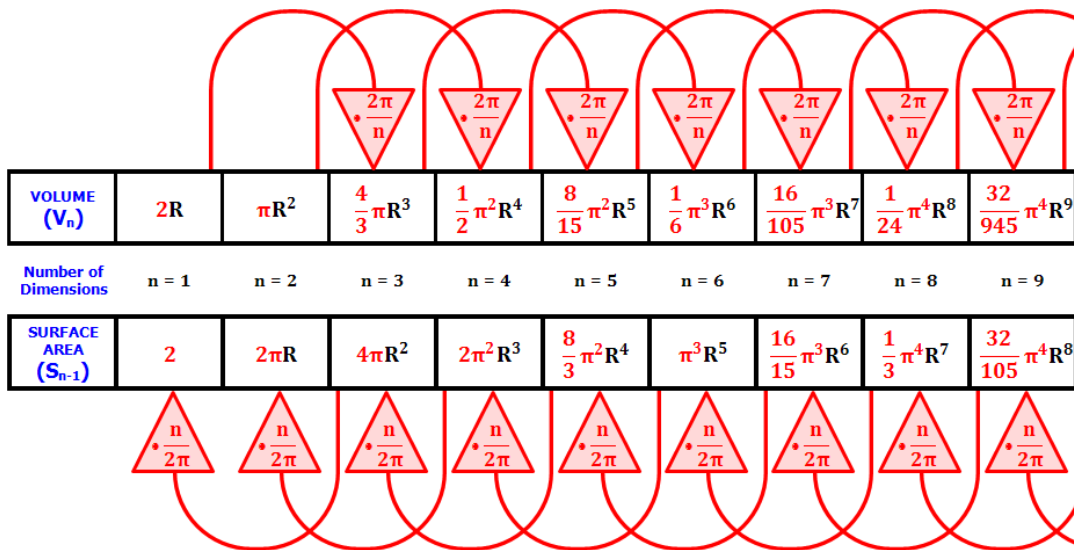
$$0.9929273350 \qquad \textbf{(7)}$$

By now you should have noticed how easy it is to extend the number of dimensions. To prove a point, our last example will be used to find the volume of a 9-D sphere: x1^2+x2^2+x3^2+x4^2+x5^2+x6^2+ x7^2+x8^2+x9^2 < R^2. Now we'll have to generate random values for x1, x2, x3, x4, x5, x6, x7, x8 and x9 all being between [0, R] (where R = 1 in this tutorial). Keep in mind that we are now effectively evaluating a 9-D integral with a tricky restriction on the integration limits. So that we can compare our Monte Carlo simulation to expected results, we'll note that a 9-D sphere has a volume of (32/945) π^4R^9.

```
> d := 9 :
  n := 100000 :
  Zn := 0 :
  for i from 1 to n do:
  x1 := X( ) :
  x2 := X( ) :
  x3 := X( ) :
  x4 := X( ) :
  x5 := X( ) :
  x6 := X( ) :
  x7 := X( ) :
  x8 := X( ) :
  x9 := X( ) :
```
$$R := \mathrm{sqrt}\left(x1^2 + x2^2 + x3^2 + x4^2 + x5^2 + x6^2 + x7^2 + x8^2 + x9^2\right) :$$
```
     if R ≤ 1 then;
        Zn := Zn + 1 :
     end if;
  end do:
```
$$V9D := evalf\left(\frac{2^d \cdot Zn}{n}\right);$$

$$evalf\left(\frac{V9D}{\frac{32}{945}\pi^4}\right);$$

$$V9D := 3.374080000$$
$$1.022910684 \qquad \textbf{(8)}$$

Here is a table from Wikipedia that shows the volume and surface areas of n-dimensional spheres from $n = 1$ to $n = 9$.

	n = 1	n = 2	n = 3	n = 4	n = 5	n = 6	n = 7	n = 8	n = 9
VOLUME (V_n)	$2R$	πR^2	$\frac{4}{3}\pi R^3$	$\frac{1}{2}\pi^2 R^4$	$\frac{8}{15}\pi^2 R^5$	$\frac{1}{6}\pi^3 R^6$	$\frac{16}{105}\pi^3 R^7$	$\frac{1}{24}\pi^4 R^8$	$\frac{32}{945}\pi^4 R^9$
SURFACE AREA (S_{n-1})	2	$2\pi R$	$4\pi R^2$	$2\pi^2 R^3$	$\frac{8}{3}\pi^2 R^4$	$\pi^3 R^5$	$\frac{16}{15}\pi^3 R^6$	$\frac{1}{3}\pi^4 R^7$	$\frac{32}{105}\pi^4 R^8$

Triangle multipliers (top, pointing down): $\cdot \frac{2\pi}{n}$

Triangle multipliers (bottom, pointing up): $\cdot \frac{n}{2\pi}$

"N SpheresVolumeAndSurfaceArea" by Joseph Lindenberg - Own work. Licensed under CC BY-SA 3.0 via Wikimedia Commons - http://commons.wikimedia. org/wiki/File:N_SpheresVolumeAndSurfaceArea.png#/media/File:N_SpheresVolumeAndSurfaceArea. png